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(i, j) competition graphs

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Abstract

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If D is an acyclic digraph, its competition graph has the same vertex set as D and an edge between vertices x and y if and only if for some vertex u, there are arcs (x,u) and (y,u) in D. We study competition graphs of acyclic digraphs D when the indegrees and outdegrees of the vertices of D are restricted. Under degree restrictions, we characterize the competition graphs and are able to solve the important open problem of characterizing acyclic digraphs whose competition graphs are interval graphs. We also characterize the competition graphs which are interval graphs.

1. Introduction

Suppose D is an acyclic digraph. The *competition graph* $G(D)$ of D has the same set of vertices as D and an edge between vertices x and y iff there is a vertex u such that (x,u) and (y,u) are arcs of D. Competition graphs arose in the work of Cohen [5] on competition between species in ecosystems. Here, D is a *food web* whose ver-

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tices are species in the ecosystem and which has an arc (x, u) iff x preys on u. There is an edge $\{x, y\}$ in $G(D)$ iff x and y have a common prey in the food web. Cohen [5–7] observed empirically that (almost all) competition graphs of acyclic digraphs representing food webs are interval graphs. A graph $G=(V, E)$ is an *interval graph* if we can assign to each x in V a real interval $J(x)$ so that whenever $x \neq y$,

$$\{x, y\} \in E \Leftrightarrow J(x) \cap J(y) \neq \emptyset.$$

Roberts [34] asked if Cohen's empirical observation was just an artifact of the construction. He showed that it wasn't, by showing that every graph can be made into a competition graph by adding isolated vertices. He then asked for a characterization of competition graphs of acyclic digraphs and, along with Cohen [7] and Roberts [33], for a characterization of acyclic digraphs D whose competition graphs $G(D)$ are interval graphs. We shall call such digraphs *interval digraphs*. (The term interval digraph is used in another sense by Kabell [20].) The problem of characterizing interval digraphs has remained elusive and it is, in our opinion, the basic open problem in the study of competition graphs.

Early results on the first problem, that of characterizing competition graphs of acyclic digraphs, were obtained by Roberts [34] and by Opsut [31]; the latter showed that recognition of competition graphs is an NP-complete problem. However, useful characterizations were subsequently obtained by Dutton and Brigham [13] and Lundgren and Maybee [26]. Dutton and Brigham [13] also obtained characterizations of competition graphs of arbitrary digraphs (loops allowed) and Roberts and Steif [38] obtained characterizations of competition graphs of arbitrary digraphs without loops. Harary, Kim, and Roberts [17] studied an extremal problem arising from the problem of characterizing competition graphs of acyclic digraphs and Kim and Roberts [23] studied a conjecture arising from Opsut's [31] paper. Raychaudhuri and Roberts [32] pointed out that the notion of competition graph has applications outside of ecology (in particular to communication over a noisy channel, channel assignments for radio and television transmitters, and modeling complex systems). Using an idea introduced by Roberts [35, 37], they studied a generalized notion of competition graph and began the study of competition graphs when special assumptions were made about the digraphs (in their case that the digraphs were symmetric 1-unit sphere graphs). This work is continued by Lundgren, Rasmussen, and Maybee [28, 29]. Other authors have introduced and studied variations on ordinary competition graphs. These include the common enemy graph (resource graph) studied by Lundgren and Maybee [27] and Sugihara [43]; the competition-common enemy graph studied by Scott [40], Jones et al. [19], Seager [41], and Kim, Roberts and Seager [24]; the niche graph studied by Cable et al. [3] and Bowser and Cable [1]; and the p-competition graph studied by Isaak et al. and Kim et al. [18, 22]. For surveys of the literature of competition graphs, see [21, 25, 32].

In the meantime, some progress was made on the second problem, that of characterizing those acyclic digraphs whose competition graphs are interval graphs. Cohen [7] approached this problem from a statistical point of view, trying to build statistical models for the construction of D so that $G(D)$ is (likely to be) an interval graph. Steif [42] showed that there could be no forbidden subgraph characterization of interval digraphs. Lundgren and Maybee [27] gave some results which characterize when D is an interval digraph. But these results essentially boil down to calculating $G(D)$ and using one of the well-known (and efficient) characterizations of when a given graph is an interval graph. While this solves the problem, it is not what we want: A characterization in terms of properties of D.

Since the general problem of characterizing interval digraphs seems difficult, it occurred to us that it might be reasonable to attack it under various assumptions about the digraph D. The type of assumption which is explored here is a constraint on both the indegrees and outdegrees in D. Assumptions which limit the number of predators or prey of a species seem reasonable from the point of view of the original Cohen application. Empirical results of Cohen and Briand [8] suggest that the total number of arcs per species in a food web is actually quite small in an *average* sense, i.e., it is about 2. (For other relevant empirical data, see [2]. For a random digraph model developed to account for such data, see [9–11, 30].) Assumptions which limit the indegree or outdegree of a vertex also seem reasonable from the point of view of the other applications of competition graphs explored by Raychaudhuri and Roberts [32].

We say that an acyclic digraph

D is an	if for every vertex x
(i,j) *digraph*	$\mathrm{id}(x)\le i$ and $\mathrm{od}(x)\le j$;
$(\bar{i},\bar{j})$ *digraph*	$\mathrm{id}(x)=0$ or i and $\mathrm{od}(x)=0$ or j;
$(\bar{i},j)$ *digraph*	$\mathrm{id}(x)=0$ or i and $\mathrm{od}(x)\le j$;
$(i,\bar{j})$ *digraph*	$\mathrm{id}(x)\le i$ and $\mathrm{od}(x)=0$ or j.

We say that a graph G is a (u,v) *competition graph*, where $u=i$ or $\bar{i}$ and $v=j$ or $\bar{j}$, if it is the competition graph of a (u,v) digraph. We say it is a (u,v) *interval competition graph* if it is a (u,v) competition graph which is also an interval graph. In this paper, we shall study three problems:

(1) Characterize the (u,v) competition graphs.
(2) Characterize the (u,v) interval competition graphs.
(3) Characterize the (u,v) digraphs which give rise to interval competition graphs. We call the latter (u,v) *interval digraphs.*

Section 2 studies the case where (u,v) is $(2,2)$. In light of the Cohen–Briand empirical results referred to above, this case is a reasonable first approximation. Section 3 studies the general case where $(u,v)=(\bar{i},\bar{j})$, $i,j\ge 2$. In that section, we add the additional special assumption that the digraph never has all four arcs (x,u), (x,v), (y,u), (y,v). This assumption has a natural ecological interpretation (see below). Although the assumption that every species has *exactly* 0 or i predators and *exactly*

0 or j prey is rather special, we felt that it was again a reasonable special case with which to start, and it led to some interesting results. The final section, Section 4, mentions open problems.

We shall adopt the graph-theoretical terminology of Roberts [36] except that the terms path and cycle here replace the terms chain and circuit used by Roberts. If (x,u) is an arc of digraph D, we shall use the terminology that x *preys* on u or *eats* u and that x is a *predator* of u and u is a *prey* of x. Also, if $\{x,y\}$ is an edge of $G(D)$, we shall say that x and y *compete*. Finally, all digraphs in this paper will be acyclic unless explicitly mentioned otherwise.

2. (2,2) competition graphs

In this section, we study the case (2,2). Assuming that each indegree and outdegree is bounded by two is not an unreasonable first assumption: Each species has at most two predators and at most two prey. We begin by recalling the competition number of a graph. If G is any graph, Roberts [34] proved that $G \cup I_k$ is a competition graph for sufficiently large k, where $G \cup I_k$ stands for G together with k isolated vertices. The smallest such k is called the *competition number* of G and is denoted by $k(G)$.

Lemma 2.1. *Suppose every connected component of a graph G is a cycle or a path of length one or more. Then the competition number of G is* 2 *if every component is a cycle of length* > 3, *and it is* 1 *otherwise.*

Proof. By the results of Roberts [34], the competition number of Z_n, the cycle of length n, is 1 if $n=3$ and 2 if $n>3$, and the competition number of P_n, the path of length $n-1$, is 1 for $n>1$. Suppose some component is not a cycle of length > 3. Order the components of G as $K^1, K^2, \ldots, K^q$ by first listing all cycles of length > 3, then all triangles, and then all paths. Let D^i be a food web for $K^i \cup I_{k(K^i)}$. By the construction in [34], when K^i is a cycle or a path, D^i can be taken to be a (2,2) digraph with two vertices of indegree equal to 0. Build a food web by taking the disjoint union of all the D^i. Modify this to obtain a food web D as follows. If K^p is Z_n, $n>3$, replace the two new isolated vertices added to Z_n by two vertices of indegree equal to 0 in D^{p+1}. If K^p is Z_3 or P_n and $p \neq q$, replace the isolated vertex added to K^p by one vertex of indegree 0 in D^{p+1}. Finally, one isolated vertex remains added to K^q. This shows that the competition number is at most 1. It is exactly 1 since every competition graph has an isolated vertex. If every component of G is a cycle of length > 3, then by the results of Roberts [34], the competition number of each component is 2. Then build a food web D by using the same construction as above, replacing the two isolated vertices added to K^p, $p \neq q$, by two vertices of indegree equal to 0 in D^{p+1}. Leave the two isolated vertices added to K^q. This shows that the competition number is at most 2. That $k(G)$ is at least 2 follows by the result of Roberts [34] that for a triangle-free graph, the competition number is at least the number of edges minus the number of vertices plus 2. □

Remark. We remark for later use that the food webs D constructed in the proof of Lemma 2.1 are all $(2,2)$ digraphs. If no component is a path of length 2 or more, then all these food webs are $(\bar{2},\bar{2})$ digraphs.

The graph $K_{1,3}$ consists of a vertex x and three nonadjacent neighbors of x.

Lemma 2.2. *A* $(2,2)$ *competition graph has no generated* $K_{1,3}$.

Proof. Suppose that in a $(2,2)$ competition graph G, there are vertices x, a, b, and c so that x competes with a, b, and c, none of which compete with each other. Then there are α, β, and γ so that x and a prey on α, x and b prey on β, and x and c prey on γ. Since a, b, and c do not compete, α, β, and γ are all different. It follows that $\mathrm{od}(x)>2$, which is a contradiction. □

Lemma 2.3. *If a* $(2,2)$ *competition graph* G *has a triangle, that triangle is a connected component of* G.

Proof. Suppose $G=G(D)$ for $(2,2)$ digraph D. Suppose vertices a, b, and c form a triangle in G. Then a, b, and c cannot have one common prey x, because $\mathrm{id}(x)\le 2$. Thus there are α, β, γ, all different, so that a and b prey on α, b and c prey on β, and a and c prey on γ. Then no other x can compete with a, b, or c, because in D each of a, b, and c cannot have $\mathrm{od}>2$ and so have no prey other than α, β, γ, and each of α, β, γ cannot have $\mathrm{id}>2$ and so have no predators other than a, b, or c. It follows that the triangle a,b,c is a component of G. □

Our first theorem characterizes the $(2,2)$ competition graphs.

Theorem 2.4. *A graph is a* $(2,2)$ *competition graph if and only if each connected component is an isolated vertex, a path, or a cycle, and the number of isolated vertices is at least* 2 *if every connected component is a cycle of length* >3 *and at least* 1 *otherwise.*

Proof. $(\Leftarrow)$ This follows from the proof of Lemma 2.1, for by the Remark after the lemma, the digraphs constructed are all $(2,2)$ digraphs.

$(\Rightarrow)$ Suppose a connected component K of G has a cycle in it. Then either K is a triangle, in which case K is a cycle, or, by Lemma 2.3, K has no triangles. Suppose K has no triangles. By Lemma 2.2, K has no generated $K_{1,3}$. Thus each vertex of K has degree exactly two, and it follows that K is a cycle.

Suppose next that K has no cycles in it. Then K is either an isolated vertex or a tree. But a tree with no $K_{1,3}$ is a path.

Finally, the number of isolated vertices required follows from Lemma 2.1. □

We shall now turn to interval graphs. The reader is referred to Fishburn [14], Golumbic [15] or Roberts [33] for a summary of characterizations of interval graphs. We shall use the following properties of interval graphs in this paper:

(1) If G is an interval graph, then G has no Z_n, $n>3$, as a generated subgraph.

(2) Paths, triangles, and single vertices are interval graphs.

(3) G is an interval graph if and only if every component of G is an interval graph.

(4) Every interval graph has a *simplicial vertex*, a vertex whose neighborhood is a clique.

From Theorem 2.4 we can derive a characterization of $(2,2)$ interval competition graphs.

Corollary 2.5. *A graph is a* $(2,2)$ *interval competition graph if and only if each connected component is an isolated vertex, a path, or a triangle, and the number of isolated vertices is at least* 1.

Proof. A cycle Z_n, $n>3$, cannot appear in an interval graph. But paths and triangles are interval graphs and a graph is an interval graph if and only if every component is. □

Although we study $(\bar{i},\bar{j})$ competition graphs in the next section, it is useful to observe that the $(\bar{2},\bar{2})$ competition graphs can be characterized analogously to the characterization of $(2,2)$ competition graphs given in Theorem 2.4.

Theorem 2.6. *A graph is a* $(\bar{2},\bar{2})$ *competition graph if and only if each connected component is an isolated vertex, a path of length* 1, *or a cycle, and the number of isolated vertices is at least* 2 *if every connected component is a cycle of length* >3 *and at least* 1 *otherwise.*

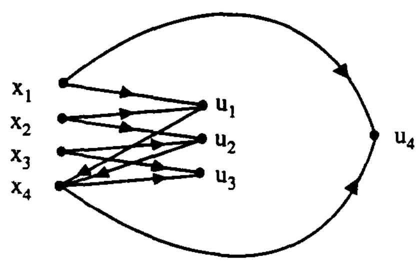

Fig. 1. *D.*

Proof. ($\Leftarrow$) This follows from the proof of Lemma 2.1, for by the Remark after the lemma, the digraphs constructed are all $(\bar{2},\bar{2})$ digraphs as long as G has no component which is a path of length 2 or more.

($\Rightarrow$) Suppose a connected component K is a path $x_1, x_2, \ldots, x_n$. Then there is a vertex u such that x_1, x_2 prey on u. Since $\mathrm{od}(x_1)>0$, there is a vertex $v \neq u$ so that x_1 preys on v. Since $\mathrm{id}(v)>0$, there is $y \neq x_1$ so that y preys on v. But since K is a path, y must be x_2. Since D is a $(\bar{2},\bar{2})$ digraph, K is the path x_1, x_2. The result now follows by Theorem 2.4. □

Corollary 2.7. *A graph is a* $(\bar{2},\bar{2})$ *interval competition graph if and only if each component is an isolated vertex, a path of length* 1, *or a triangle, and there is at least one isolated vertex.*

Steif [42] has observed that there is no forbidden subgraph characterization of interval digraphs. However, we do have the following result.

Theorem 2.8. *There is a forbidden subgraph characterization of* (2,2) *interval digraphs.*

Proof. It suffices to show that if $G(D')$ is not an interval graph and D' is a generated subgraph of a (2,2) digraph D, then $G(D)$ is not an interval graph. If $G(D')$ is not an interval graph, then by Theorem 2.4, $G(D')$ has a generated Z_n, $n>3$. Therefore $G(D)$ has a cycle of length >3. By Theorem 2.4, $G(D)$ has a generated Z_n, $n>3$. □

The forbidden subgraph characterization which results from the proof of Theorem 2.8 is not very useful: It is necessary to list all digraphs D so that $G(D)$ is not an interval graph, i.e., so that $G(D)$ has Z_n, $n>3$, as some component. It is not good enough to list those giving rise to $Z_n \cup I_r$. To see why, note that the digraph D of Fig. 1 has in $G(D)$ one 4-cycle, one path of length 1, and two isolated vertices. But there is no generated subgraph D' of D so that $G(D')$ is $Z_n \cup I_r$, some $n>3$.

In what follows, there will be occasion to talk about the digraph of Fig. 2. We shall call this digraph $P(2,2)$. Note that forbidding $P(2,2)$ as a subgraph (not only

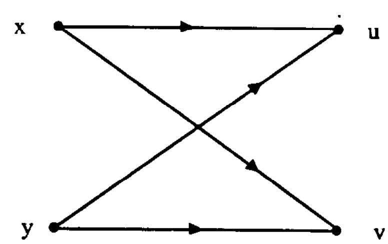

Fig. 2. $P(2,2)$.

as a generated subgraph) says that any two species can have at most one common prey. If D has no $P(2,2)$ subgraph, we say D is *irredundant*.

Lemma 2.9. *Suppose D is a $(\bar{2},2)$ digraph. Then every component of $G(D)$ is a cycle or an isolated vertex if and only if D is a $(\bar{2},\bar{2})$ irredundant digraph.*

Proof. ($\Leftarrow$) We shall show that every vertex x of $G(D)$ has degree 0 or 2. It follows from Theorem 2.6 that every component of $G(D)$ is an isolated vertex or a cycle.

If $\mathrm{od}(x)=0$, then $\deg(x)=0$ in $G(D)$. Thus, suppose that $\mathrm{od}(x)=2$. Let x prey on u and v. Since every vertex of D has indegree 0 or 2, it follows that there are y and z so that y preys on u and z preys on v. Also, $y \neq z$ since $P(2,2)$ is not a subgraph of D. It follows that $\deg(x) \geq 2$ in $G(D)$. That $\deg(x)=2$ follows since x can have no other prey and u and v no other predators.

($\Rightarrow$) Suppose $\mathrm{od}(x)=1$ for some x, and suppose x preys on u. Since D is a $(\bar{2},2)$ digraph, there is $y \neq x$ so that y preys on u. Now no other z preys on u since $\mathrm{id}(u)=2$ and x does not prey on any other vertex. Thus, $\deg(x)=1$ in $G(D)$. By Theorem 2.4, some component of x is a path, which is a contradiction. We conclude that D is a $(\bar{2},\bar{2})$ digraph.

Suppose $P(2,2)$ is a subgraph of D. Then in $G(D)$, the vertices x and y of Fig. 2 form a connected component which is a path. □

Theorem 2.10. *Suppose D is a $(2,2)$ digraph. Then $G(D)$ has a cycle if and only if D has a $(\bar{2},\bar{2})$ subgraph D' (not necessarily generated) so that D' is irredundant and has at least one arc.*

Proof. ($\Rightarrow$) Suppose $x_1, x_2, \ldots, x_n$ is a cycle in $G(D)$. Then there are $u_1, u_2, \ldots, u_n$ so that

$$\begin{aligned} &x_1, x_2 \text{ prey on } u_1,\\ &x_2, x_3 \text{ prey on } u_2,\\ &\quad\vdots\\ &x_n, x_1 \text{ prey on } u_n. \end{aligned} \tag{1}$$

Note that the x_i are distinct by definition and the u_i are distinct since none can have indegree more than 2. However, some u_i could equal some x_j. Define D' by using the vertices $x_1, \ldots, x_n$, $u_1, \ldots, u_n$ and using the arcs in (1). Now in D', $\mathrm{od}(x_i)=2$ for all i and $\mathrm{od}(y)=0$ if $y \neq x_1, \ldots, x_n$. Moreover, $\mathrm{id}(u_i)=2$ and $\mathrm{id}(v)=0$ if $v \neq u_1, \ldots, u_n$. It follows that D' is a $(\bar{2},\bar{2})$ subgraph. Moreover, D' certainly has an arc. If $P(2,2)$ of Fig. 2 is a subgraph of D', then $x=x_i$, $y=x_j$, $u=u_r$, and $v=u_s$, some $i \neq j$, $r \neq s$. If + is interpreted as modulo n, it follows that $\{r,s\}=\{i,i+1\}$ and $\{r,s\}=\{j,j+1\}$. Hence, $i=j$, which is a contradiction.

($\Leftarrow$) Suppose D' is as in the theorem. By Lemma 2.9, $G(D')$ has no components which are paths. Since D' has an arc, it has a vertex of indegree 2. Thus, $G(D')$ has an edge. It follows that some component of $G(D')$ is a cycle. Therefore $G(D)$ has a cycle. □

The following theorem gives a characterization (not a forbidden subgraph characterization) of (2, 2) interval digraphs.

Theorem 2.11. *Suppose D is a $(2,2)$ digraph. Then D is an interval digraph if and only if every $(\bar{2},\bar{2})$ irredundant subgraph of D with at least one arc contains one of the three digraphs S, T, and U in Fig. 3 as a generated subgraph.*

Proof. ($\Leftarrow$) Suppose D is not an interval digraph. By Theorem 2.4, $G(D)$ has a cycle of length greater than 3 as a connected component. Suppose $x_1, x_2, \ldots, x_n$, $n>3$, are the vertices of this cycle. Define $u_1, u_2, \ldots, u_n$ as in (1), and define D' as in the proof of Theorem 2.10. Then D' is a $(\bar{2},\bar{2})$ subgraph with at least one arc and no $P(2,2)$ subgraph. Since $G(D')$ is Z_n plus isolated vertices, $G(D')$ has no triangles and D' cannot contain S, T, or U as a generated subgraph.

($\Rightarrow$) Suppose that D is a (2, 2) interval digraph and D' is a $(\bar{2},\bar{2})$ irredundant subgraph of D with at least one arc. Then it follows from Theorem 2.10 that $G(D')$ has a cycle $x_1, x_2, \ldots, x_n$. Hence, as in the proof of Theorem 2.10, there are $u_1, u_2, \ldots, u_n$ in D' so that equation (1) holds. Since each vertex x_i already has outdegree 2, x_i and x_j have a common prey in D if and only if $|i-j| \equiv 1 \pmod{n}$. Thus, $x_1, x_2, \ldots, x_n$ is a generated cycle in $G(D)$. Since $G(D)$ is an interval graph, $n=3$. If $x_i \neq u_j$, all i and j, and there are no other arcs in the subgraph generated by x_1, x_2, x_3, u_1, u_2, u_3, then these vertices define a generated subgraph of the form of digraph S of Fig. 3. If there are additional arcs in the subgraph, then the only possibility is that u_i preys on x_j for some i and j, since D is a (2, 2) digraph. In this case, x_j cannot prey on u_i and we have the digraph U of Fig. 3 as a subgraph. This subgraph is generated because the presence of a second arc (u_s, x_t) creates a cycle. If $x_i = u_j$ for some i and j, then the vertices define digraph T of Fig. 3 and there can be no additional arcs without forming a cycle in D or violating indegree ≤ 2 or outdegree ≤ 2. □

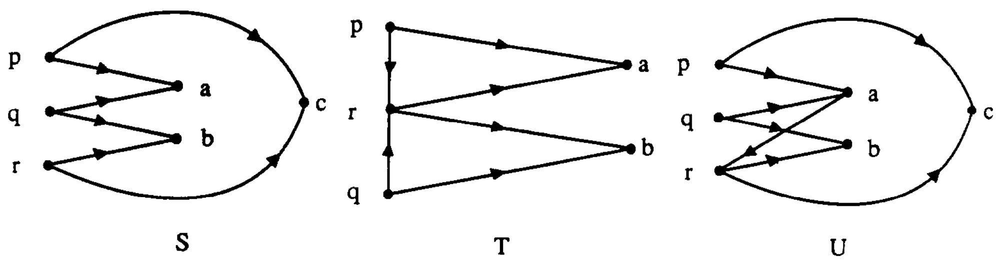

Fig. 3.

We say that vertices $x \neq y$ of D are *triangulated* in D if there is a subgraph of D of type S or T of Fig. 3 with x and y in $\{p, q, r\}$. (Note that if there is a subgraph of type U, then there is a subgraph of type S.)

Theorem 2.12. *Suppose D is a $(2,2)$ digraph with n vertices. Then $G(D)$ is an interval graph if and only if the vertices of D can be labelled $1, 2, \ldots, n$ so that if i and j are triangulated, then $|i-j| = 1$ or 2 and if i and j have a common prey but are not triangulated, then $|i-j| = 1$.*

Proof. ($\Rightarrow$) List all isolated vertices of $G(D)$ first, then all components which are paths, and then all components which are cycles and therefore triangles. (Cf. Theorem 2.4.) First label vertices in the first component in order along a path or a cycle, then label vertices in the second component, and so on. The labeling has the property that competing vertices get labels within 1 if they are on a path and within 2 if they are on a triangle. Finally, the only way that i and j can be on a triangle is if they are triangulated in D.

($\Leftarrow$) If $G(D)$ has a component Z_n, $n > 3$, this component and therefore $G(D)$ cannot be labelled because no two of the vertices in the component are triangulated. □

The following theorem has a similar proof.

Theorem 2.13. *Suppose D is a $(2,2)$ digraph with n vertices. Then $G(D)$ has no cycles if and only if the vertices of D can be labelled $1, 2, \ldots, n$ in such a way that if i and j have a common prey, then $|i-j| = 1$.*

We now describe an algorithm which, when applied to a $(2,2)$ digraph D, can be used to recognize whether or not D is an interval digraph. The heart of the algorithm is a subroutine R which picks a vertex x_1 of D at random and keeps identifying vertices in the component of x_1 in $G(D)$ until either a cycle is found or we find that there is a vertex of degree 1 in the component and hence the component is not a cycle. The subroutine R can be outlined as follows.

Subroutine R.

Step 1. Pick x_1 from the set of vertices not yet used in any previous run of this subroutine. If there is no such x_1, terminate (the subroutine) with the message "D is an interval digraph".

Step 2. Find a prey u_1 of x_1. If there is no u_1, terminate with the message "continue".

Step 3. Find a predator $x_2 \neq x_1$ of u_1. If there is no such predator, terminate with the message "continue".

Step 4. Find a prey $u_2 \neq u_1$ of x_2. If there is no such prey, terminate with the message "continue".

Step 5. Having found $x_1, x_2, \ldots, x_p$, $u_1, u_2, \ldots, u_p$, find a predator $x_{p+1} \neq x_p$ of u_p. If there is no such predator, terminate with the message "continue". If $x_{p+1} = x_1$ and $p+1=3$, terminate with the message "continue". If $x_{p+1} = x_1$ and $p+1>3$, terminate with the message "D is not an interval digraph".

Step 6. Find a prey $u_{p+1} \neq u_p$ of x_{p+1} and repeat Step 5 with $p=p+1$. If there is no such prey, terminate with the message "continue".

Based on this subroutine, we now describe the algorithm.

Algorithm A.
Repeat Subroutine R whenever it terminates with the message "continue". Stop when it terminates with either the message "D is an interval digraph" or "D is not an interval digraph".

Theorem 2.14. *Given a* $(2,2)$ *digraph* D, *Algorithm* A *always terminates and it terminates with the message "D is an interval digraph" if and only if this is the case. The complexity of the algorithm is* $O(n+a)$, *where* n *is the number of vertices of* D *and* a *the number of arcs.*

Proof. Note that Subroutine R terminates at Step 1 when all components have been studied and we did not find a component which was a cycle of length > 3. It terminates at Step 2 when x_1 is isolated and therefore we continue to look for other components which are cycles of length > 3. If it terminates at Step 3, then x_1 has at most one other prey and therefore either x_1 is isolated or the component containing x_1 is a path. We therefore continue to look for components which are cycles of length > 3. The subroutine terminates at Step 4 if x_2 has degree 1 in $G(D)$ and therefore the component containing x_1 is a path. Again, we continue looking for a component which is a cycle of length > 3. If R terminates at Step 5 with the message "continue", then either we could not find a predator x_{p+1}, in which case x_p has degree 1 and therefore the component containing x_1 is a path; or $x_{p+1} = x_1$ and $p+1=3$, in which case we have found a cycle of length 3 and therefore this is the component containing x_1. We still continue looking for a component which is a cycle of length > 3. If R terminates at Step 5 with the message "D is not an interval digraph", then we have found a cycle of length > 3 and this, by Theorem 2.4, is in fact a component of $G(D)$. Hence, $G(D)$ is not an interval graph. Finally, R terminates at Step 6 if x_{p+1} has degree 1 and again the component containing x_1 is a path. We now continue looking for other components which might be cycles of length > 3. Once x_1 is chosen to start Subroutine R, we never start the subroutine with x_1 again. Thus, we are sure the subroutine will be used at most n times.

The algorithm can be implemented so that each vertex of D is visited at most three times and each arc of D is investigated at most once. To make sure of the latter, we simply discard an arc once it is used. To make sure of the former, we simply keep a record of whether a vertex has been used before as a predator, and if so, we do not consider it again as a possible predator. Thus, each vertex is used at most once as a predator. It follows that each vertex is used at most twice as a prey since its indegree is at most 2. Thus, the complexity of the algorithm is $O(n+a)$. □

3. $(\bar{i},\bar{j})$ irredundant competition graphs

In this section, we study the competition graphs of $(\bar{i},\bar{j})$ digraphs. We assume throughout this section that $i,j \geq 2$. This assumption simply eliminates the uninteresting case of vertices of outdegree or indegree 1, vertices which play no role in determining competition. We shall limit our discussion to irredundant digraphs D, those which have no subgraph $P(2,2)$. Irredundancy is a natural assumption for food webs: No two species have more than one common prey, and so competition is not redundant. If G is the competition graph of an irredundant digraph, we call G an *irredundant competition graph*. In this section we study our three basic questions for $(\bar{i},\bar{j})$ irredundant digraphs. That is, we shall try to characterize the $(\bar{i},\bar{j})$ irredundant competition graphs, the $(\bar{i},\bar{j})$ irredundant interval competition graphs, and the $(\bar{i},\bar{j})$ irredundant interval digraphs.

The restriction that every species has *exactly* 0 or i predators and *exactly* 0 or j prey is certainly rather special. However, it seems like another reasonable special case with which to start. As it turns out, so much regularity leads in a natural way to the applicability of the theory of combinatorial designs. This came as a pleasant surprise to us.

Given a digraph or food web D, it will be useful in this section to use the notation

$$P(x) = \{u: x \text{ eats } u\}$$

and

$$Q(u) = \{x: x \text{ eats } u\}.$$

Throughout this section, it will also be useful to use concepts of combinatorial designs. All of our terminology follows Dembowski [12] or Hall [16]. We begin by reminding the reader of the relevant concepts.

A *tactical configuration* is a collection of b sets called *blocks* which are all subsets of a v-element set whose members are called *varieties*, with the requirements that $b>0$, $v>0$,

(a) each block has the same size, $k>1$,
(b) each variety appears in the same number $r>0$ of blocks.

A tactical configuration is called a *balanced incomplete block design* or BIBD if in addition

(c) every pair of varieties appears in common in the same number $\lambda>0$ of blocks.

A BIBD satisfying conditions (a)–(c) is also called a (b,v,r,k,λ)-*design*.

Given a tactical configuration, suppose we partition the 2-element subsets of the set of varieties into classes $A_0, \ldots, A_{t-1}$ in such a way that

(d) if $\{x,y\} \in A_i$, then the number of blocks containing both x and y is a number λ_i independent of x and y.

In this case, we call such a configuration a *tactical configuration with a consistent t-class scheme*. If $t=2$ and $\lambda_0=0$ and $\lambda_1=1$, we call the tactical configuration a *mixed 2-design* or a (b,v,r,k) *mixed 2-design*. The sets A_i in a tactical configuration with a consistent t-class scheme can be empty. If $A_0=\emptyset$ in a (b,v,r,k) mixed 2-design, then we have a $(b,v,r,k,1)$-design.

Given a tactical configuration with a consistent t-class scheme, we call it a *partially balanced incomplete block design* or PBIBD if it satisfies the additional condition:

(e) if $\{x,y\} \in A_h$, then the number of z such that $\{x,z\} \in A_r$ and $\{y,z\} \in A_s$ depends only on h, r, and s and not on x and y.

A partition of 2-element subsets of a set which satisfies condition (e) (but not necessarily condition (d)) is called an *association scheme*.

Note that there exist $(\bar{i},\bar{j})$ irredundant competition graphs for every $i,j \geq 2$. For every digraph with no arcs is trivially an $(\bar{i},\bar{j})$ irredundant digraph. We shall be interested in the existence of *nontrivial* $(\bar{i},\bar{j})$ irredundant competition graphs, where a graph is called *trivial* if it has no edges.

Theorem 3.1. *Suppose $i,j \geq 2$. Then there is a nontrivial $(\bar{i},\bar{j})$ irredundant competition graph if and only if there is a (b,v,r,k) mixed 2-design with parameters $r=j$ and $k=i$.*

Proof. ($\Leftarrow$) Suppose P is such a mixed 2-design. Define D by letting $V(D)$ be the set of blocks of P plus the set of varieties. Let all elements of block B_u prey on block B_u. We denote this digraph by $D(P)$. Clearly D is acyclic. Moreover, there is no $P(2,2)$ since we cannot have $\{x,y\} \in B_u \cap B_v$ for $u \neq v$. The indegree of each variety is 0 and the indegree of each block is $k=i$. The outdegree of each block is 0 and the outdegree of each variety is $r=j$. Since $k=i \geq 2$, there are competing vertices in each block. It follows that $G(D)$ is a nontrivial $(\bar{i},\bar{j})$ irredundant competition graph. For future reference, we denote this graph by $G(P)$.

($\Rightarrow$) Suppose $G=G(D)$ is an irredundant $(\bar{i},\bar{j})$ competition graph. Define the tactical configuration $P=P(D)$ by letting the varieties be all the non-isolated vertices of G and letting the blocks be all the sets $Q(u)$ which are nonempty. Then P defines a (b,v,j,i) mixed 2-design. For each block has size $i=k>1$ since each u has indegree equal to 0 or i and the nonempty $Q(u)$ correspond to vertices u of indegree i. There is a block and there is a variety since G is nontrivial. Moreover, each variety or non-isolated vertex x has outdegree equal to j in D and hence appears in j blocks $Q(u_1), \ldots, Q(u_j)$, where $P(x)=\{u_1, \ldots, u_j\}$. Note that these blocks are distinct since D has no $P(2,2)$. Moreover, $r=j>0$. Suppose x and y are two different varieties.

If they don't compete, then x and y appear together in no block. If they do compete, then because D has no $P(2,2)$'s, x and y appear together in exactly one block. $\square$

Note that PBIBDs with two classes with $\lambda_0=0$ and $\lambda_1=1$, which are special cases of mixed 2-designs, exist for many $k=i$, $r=j$. They are tabulated in [4].

Corollary 3.2. *Suppose $i,j\geq 2$ and there is a (b,v,r,k,λ)-design with $r=i$, $k=j$, and $\lambda=1$. Then there is a nontrivial $(\bar{i},\bar{j})$ irredundant competition graph.*

Proof. Suppose P is a $(b,v,i,j,1)$-design. Build a new tactical configuration P^* by letting the blocks of P become the varieties of P^* and the varieties of P become the blocks of P^*. Specifically, in P^*, place block B_u in variety x iff in P, variety x is in block B_u. It is easy to see that P^* is a (v,b,j,i) mixed 2-design. The result now follows from Theorem 3.1. $\square$

Remark 3.3. This corollary allows us to show that there are nontrivial $(\bar{i},\bar{j})$ irredundant competition graphs with $i>j$. Indeed, there is one whenever there is a $(b,v,i,j,1)$-design. By contrast, we shall show below that there are no nontrivial $(\bar{i},\bar{j})$ irredundant *interval* competition graphs when $i>j$. To show that there can be nontrivial $(\bar{i},\bar{j})$ irredundant competition graphs with $i>j$, consider the $(6,4,3,2,1)$-design consisting of all pairs from a 4-element set. Let P be the corresponding $(4,6,2,3)$ mixed 2-design obtained as in the proof of Corollary 3.2 by interchanging varieties and blocks. Then the graph $G(P)$ constructed in the proof of Theorem 3.1 is a $(\bar{3},\bar{2})$ irredundant competition graph. This graph consists of four isolated vertices plus the complement of the graph H consisting of three disjoint edges.

Note that Theorem 3.1 does not answer our first question, i.e., it does not characterize the graphs which are $(\bar{i},\bar{j})$ irredundant competition graphs. We do not have a complete answer to this question. The next few theorems give some information about such graphs.

Theorem 3.4. *If $i,j\geq 2$ and G is an $(\bar{i},\bar{j})$ irredundant competition graph, then every nontrivial component of G is $j(i-1)$-regular.*

Proof. Suppose x is not isolated in $G=G(D)$. In D, x has prey $u_1,\dots,u_j$. Each of the u_i has $i-1$ predators different from x. Let $R_p=Q(u_p)-\{x\}$. If $R_p\cap R_q\neq\emptyset$ for some $p\neq q$, then there is $y\in R_p\cap R_q$ such that both y and x prey on u_p and u_q. This cannot be, for otherwise D would have $P(2,2)$. It follows that $\bigcup_{p=1}^{j} R_p$ has exactly $j(i-1)$ elements in it, and these elements are exactly those which compete with x in G. $\square$

The next theorem shows that the converse of Theorem 3.4 is essentially true when $i=2$.

Theorem 3.5. *Suppose $j \geq 2$ and every nontrivial component of G is j-regular. Then G plus sufficiently many isolated vertices is a $(\bar{2}, \bar{j})$ irredundant competition graph.*

Proof. Suppose D is defined from G by taking the vertices of G, adding one vertex u_{xy} corresponding to each edge $\{x,y\}$ of G, and letting x and y from edge $\{x,y\}$ prey on u_{xy}. Then G plus sufficiently many isolated vertices is the competition graph of D. Moreover, since every non-isolated vertex of G is on j edges, by j-regularity, it follows that every such vertex has outdegree j in D. All other vertices in D have outdegree 0. Finally, vertices u_{xy} have indegree 2 in D and all other vertices in D have indegree 0. Thus, D is a $(\bar{2}, \bar{j})$ digraph. Irredundancy of D is straightforward. □

Corollary 3.6. *Suppose $j \geq 2$. Then the nontrivial $(\bar{2}, \bar{j})$ irredundant competition graphs are exactly those graphs which are j-regular graphs together with sufficiently many isolated vertices.*

Proof. ($\Leftarrow$) By Theorem 3.5.

($\Rightarrow$) By Theorem 3.4, every nontrivial component of the $(\bar{2}, \bar{j})$ irredundant competition graph G is $j(i-1)$-regular $= j$-regular. Hence, it follows that G less isolated vertices is j-regular. □

The next theorem applies not only to $(\bar{i}, \bar{j})$ irredundant competition graphs, but to (i,j) irredundant competition graphs as well. The result should be contrasted with Theorem 3.8 below, which says that all $(\bar{i}, \bar{j})$ irredundant interval competition graphs are unions of components each of which is a single vertex or a clique of size $j(i-1)+1$. It will follow that such graphs cannot exist when $i > j$.

Theorem 3.7. *If $i > j \geq 2$, then no (i,j) irredundant competition graph has a clique of size $j(i-1)$.*

Proof. Note that if $i > j \geq 2$, then $j(i-1) - i = i(j-1) - j > j(j-1) - j = j(j-2) \geq 0$, so $i < j(i-1)$. Suppose $G = G(D)$, where D is an (i,j) irredundant digraph, and suppose $K = \{x_1, \ldots, x_{j(i-1)}\}$ is a clique of G. We shall reach a contradiction. Let x_r and x_s both prey on u_{rs}. We show first that there are p, q so that u_{pq} is eaten by i vertices from K. Fix p. Note that since x_p has outdegree at most j, there are at most j different u_{pq}'s. If every u_{pq} has at most $i-2$ predators from K other than x_p, then x_p competes with at most $(i-2)j$ such vertices and K isn't a clique. Thus, we can find u_{pq} with i predators from K.

Without loss of generality let $x_1 = x_p$ and let $x_1, \ldots, x_i$ prey on u_{pq}. Since $i < j(i-1)$, there is x_t in K, $t > i$. Moreover, x_t does not prey on u_{pq}. Consider u_{rt}, $r = 1, \ldots, i$. Since x_t preys on all such u_{rt} and since x_t has outdegree at most $j < i$, we have $u_{rt} = u_{st}$ for some $r \neq s$, $r, s \leq i$. Note that $u_{rt} \neq u_{pq}$ since x_t preys on u_{rt} and not u_{pq}. It follows that x_r, x_s, u_{pq}, and u_{rt} form a $P(2,2)$. This is a contradiction. □

We now turn our attention to $(\bar{i}, \bar{j})$ irredundant interval competition graphs.

Theorem 3.8. *If $i, j \geq 2$ and G is an $(\bar{i}, \bar{j})$ irredundant interval competition graph, then every nontrivial component of G is the complete graph $K_{[j(i-1)+1]}$.*

Proof. Suppose G is an $(\bar{i}, \bar{j})$ irredundant interval competition graph. By Theorem 3.4, every nontrivial component of G is regular of degree $j(i-1)$. Since G is an interval graph, each component has a simplicial vertex (see the definition in Section 2). If x is a simplicial vertex of a nontrivial component, then x has $j(i-1)$ neighbors and so the component has a clique of size $j(i-1)+1$. Thus, the component must be $K_{[j(i-1)+1]}$. □

Corollary 3.9. *If $i, j \geq 2$ and G is an $(\bar{i}, \bar{j})$ irredundant interval competition graph, then every vertex of G is a simplicial vertex.*

Theorem 3.8 gives us a necessary condition for a graph to be an $(\bar{i}, \bar{j})$ irredundant interval competition graph. We shall see below that this condition is not sufficient. However, the following theorem gives us a type of characterization of such graphs.

Theorem 3.10. *Suppose $i, j \geq 2$ and G is a nontrivial graph. Then G plus sufficiently many isolated vertices is an $(\bar{i}, \bar{j})$ irredundant interval competition graph if and only if every nontrivial component of G is $K_{[j(i-1)+1]}$ and there is a $(b, v, r, k, 1)$-design with $r=j$ and $k=i$.*

Proof. ($\Leftarrow$) Suppose every nontrivial component of G is $K_{[j(i-1)+1]}$ and P is a $(b, v, r, k, 1)$-design with $r=j$ and $k=i$. Then $G(P)=G(D(P))$ as constructed in the proof of Theorem 3.1 is an $(\bar{i}, \bar{j})$ irredundant competition graph. Moreover, since every two varieties in P are in a block, every two vertices of $G(P)$ which are varieties compete and any other vertices of $G(P)$, i.e., those which are blocks, are isolated. It follows that $G(P)$ is $K_v \cup I_b$. By a fundamental property of designs, $r(k-1)=\lambda(v-1)$, so, since $\lambda=1$, $v=j(i-1)+1$. By using enough copies of $D(P)$ and throwing in isolated vertices if needed, we build a digraph D whose competition graph is the given G plus isolated vertices. Finally, G plus isolated vertices is an interval graph.

($\Rightarrow$) Suppose G plus sufficiently many isolated vertices is an $(\bar{i}, \bar{j})$ irredundant interval competition graph of digraph D. Then Theorem 3.8 tells us that every nontrivial component is $K_{[j(i-1)+1]}$. Let $K=K_{[j(i-1)+1]}$ be such a nontrivial component. Build a tactical configuration P' from G and D by letting the varieties be all vertices in K and letting the blocks be all sets $Q(u)$ which contain an element of K. Note that each such set $Q(u)$ is in fact contained in the set of varieties, since if x and y are in $Q(u)$ and x is in K, then x and y compete in G and so y is in K. Just as one shows in the proof of Theorem 3.1 that $P(G)$ as constructed there is a (b, v, r, k) mixed

2-design with $r=j$, $k=i$, one proves the same thing for P'. Moreover, since every pair of varieties in P' compete in G, P' is a $(b,v,r,k,1)$-design. □

Corollary 3.11. *Suppose $i,j\geq 2$ and G is a nontrivial graph. Then G plus sufficiently many isolated vertices is an $(\bar{i},\bar{j})$ irredundant interval competition graph if and only if G less isolated vertices is $j(i-1)$-regular, every vertex of G is simplicial, and there is a $(b,v,r,k,1)$-design with $r=j$ and $k=i$.*

Proof. Straightforward from Theorem 3.10. □

Remark 3.12. If there is a (b,v,r,k,λ)-design, then it is well known that $bk=vr$ and that $r(k-1)=\lambda(v-1)$. In particular, with $\lambda=1$, these two conditions imply that

$$r^2k-r^2+r \text{ is divisible by } k. \tag{2}$$

Suppose r and k are given. For $k=2$, 3, or 4, it turns out that condition (2) is necessary and sufficient for the existence of a $(b,v,r,k,1)$-design for some b and v. For $k\geq 5$, it is not. However, for given $k\geq 5$, there is r_0 so that if $r\geq r_0$, then the condition is necessary and sufficient for the existence of a $(b,v,r,k,1)$-design. These results are easily obtained from the conditions which are summarized in Hall [16].

By Remark 3.12, we know that there is no $(b,v,5,3,1)$-design. It follows that K_{11} plus sufficiently many isolated vertices is never a $(\bar{3},\bar{5})$ irredundant interval competition graph. This shows that the converse of Theorem 3.8 is false.

Corollary 3.13. *There are no nontrivial $(\bar{i},\bar{j})$ irredundant interval competition graphs when $i>j$.*

Proof 1. By Fisher's inequality for (b,v,r,k,λ)-designs, $b\geq v$ and hence $r\geq k$. (See Hall [16].) □

Proof 2. Theorems 3.7 and 3.8 give the result. □

In contrast to Corollary 3.13, there can be nontrivial $(\bar{i},\bar{j})$ irredundant competition graphs when $i>j$ as long as they are not required to be interval. Indeed, by Corollary 3.2, such graphs exist whenever $(b,v,i,j,1)$-designs exist, and these can exist for many values of $i>j$. Consider for example the graph $G(P)$ constructed in Remark 3.3. Note of course that $G(P)$ has components which are not cliques, for the complement of H defined in Remark 3.3 is such a component.

The remaining corollaries are straightforward from Remark 3.12. (Corollary 3.14 also follows from Corollary 3.6.)

Corollary 3.14. *K_{j+1} plus sufficiently many isolated vertices is a $(\bar{2},\bar{j})$ irredundant interval competition graph for all $j\geq 2$.*

Corollary 3.15. *K_{2j+1} plus sufficiently many isolated vertices is a $(\bar{3},\bar{j})$ irredundant interval competition graph if and only if $j=3n$ or $3n+1$.*

Corollary 3.16. *K_{3j+1} plus sufficiently many isolated vertices is a $(\bar{4},\bar{j})$ irredundant interval competition graph if and only if $j=4n$ or $4n+1$.*

Corollary 3.17. *For all $i \geq 5$, $K_{[j(i-1)+1]}$ plus sufficiently many isolated vertices is an $(\bar{i},\bar{j})$ irredundant interval competition graph for all j sufficiently large.*

What Theorem 3.10 and its corollaries leave undecided is the question, given G, of how many additional isolated vertices are sufficient to make G plus these isolated vertices into an $(\bar{i},\bar{j})$ irredundant interval competition graph. Let $h(G)$ be the smallest number of additional isolated vertices which will suffice, if any will, and let $h(G)$ be undefined otherwise. The reader will notice the similarity between this concept and that of competition number introduced by Roberts [34]. It remains an open question to compute $h(G)$ for all graphs for which it is defined. However, we have the following bounds on $h(G)$.

Theorem 3.18. *Suppose $i, j \geq 2$ and G is $K_{[j(i-1)+1]}$. If there is a $(b, v, j, i, 1)$-design, then*

$$\frac{[j(i-1)+1]\,j}{i} - j(i-1) - 1 + i \leq h(G) \leq \frac{[j(i-1)+1]\,j}{i} - 1.$$

Proof. Suppose P is a $(b, v, j, i, 1)$-design. Then by the well-known equality $r(k-1) = \lambda(v-1)$, we have $v = j(i-1)+1$. By the equality $bk = vr$,

$$b = \frac{[j(i-1)+1]\,j}{i}.$$

We construct $D(P)$ as in the first part of the proof of Theorem 3.1. We show in the proof of Theorem 3.10 that $D(P)$ is a food web whose competition graph is $K_{[j(i-1)+1]} \cup I_b$. Choose a vertex a in I_b. Then there are i incoming arcs toward a. Since $v - i = j(i-1)+1-i = (j-1)(i-1) \geq 1$, there is a vertex x in $K_{j(i-1)+1}$ such that x does not prey on a. Obtain digraph D' from digraph $D(P)$ by replacing a by x. This does not create a cycle by the way that $D(P)$ was constructed. Moreover, it does not create a $P(2,2)$ because if u, $v \in K_{[j(i-1)+1]}$ and $c \in I_b$ and x create a $P(2,2)$ in D', then u, v, a, c create a $P(2,2)$ in $D(P)$. Note that D' is an $(\bar{i},\bar{j})$-digraph. Finally, note that $G(D')$ is $K_{[j(i-1)+1]} \cup I_{b-1}$. This shows that

$$h(G) \leq b - 1 = \frac{[j(i-1)+1]\,j}{i} - 1.$$

To obtain the lower bound, suppose that $h = h(G)$ and $G \cup I_h$ is a competition graph of the $(\bar{i},\bar{j})$ irredundant digraph D. Suppose that P' as defined in the proof of Theorem 3.10 has b blocks. Then P' is a $(b, v, j, i, 1)$-design. We also know that by definition of P', for each block B_r there is a vertex u_r of D eaten by all the vertices in B_r, and the u_r are distinct. Let U be the set of such u_r's. Altogether, then, there are b vertices in the set U. Since D is acyclic, we can label the vertices of D as $1,2,\dots$ so that if there is an arc from x to y, then x gets a lower number than y. Then vertices labelled $1,2,\dots,i$ cannot have incoming arcs because every in-degree must be i. Hence, these first i vertices are not in the set U. We know $G' = G \cup I_h - \{1,2,\dots,i\}$ contains U, so G' has at least b vertices. Now G has v vertices, since in the construction of P' we took one variety for each vertex of G. Thus, G' has $v+h-i$ vertices. It follows that $v+h-i \geq b$, and we conclude that

$$h \geq b - v + i = \frac{[j(i-1)+1]\,j}{i} - j(i-1) - 1 + i. \qquad \square$$

Remark 3.19. The bounds in Theorem 3.18 are sharp. Suppose $i=j=2$. Then G is a triangle. There is a $(3,3,2,2,1)$-design. For this design, the upper and lower bounds in Theorem 3.18 are both 2, and hence $h(G)$ equals both the lower and upper bounds. We do not know if the bounds in Theorem 3.18 are sharp for arbitrary i and j.

We turn finally to the question: What do the $(\bar{i},\bar{j})$ irredundant interval digraphs look like? Corollary 3.9 leads naturally to a characterization of these digraphs. Let us say that a vertex x in a digraph D is *di-simplicial* if whenever there are in D arcs (x,u), (y,u), (x,v), (z,v), then there are in D arcs (y,w) and (z,w) for some w. Note that x is di-simplicial in D iff x is simplicial in $G(D)$. The next theorem gives the characterization.

Theorem 3.20. *Suppose $i,j \geq 2$. Then an $(\bar{i},\bar{j})$ irredundant digraph D is interval if and only if every vertex of D is di-simplicial.*

Proof. If D is interval, then by Corollary 3.9, every vertex of $G(D)$ is simplicial and hence every vertex of D is di-simplicial. Conversely, suppose every vertex of D is di-simplicial, so every vertex of $G(D)$ is simplicial. Then consider a nontrivial component K of $G(D)$. There is a simplicial vertex x in K and, by Theorem 3.4, x has degree $j(i-1)$. Thus, K has a clique of size $j(i-1)+1$. By $j(i-1)$-regularity, K is $K_{[j(i-1)+1]}$. Thus, $G(D)$ is an interval graph. $\square$

Corollary 3.21. *If $i,j \geq 2$ and $D=(V,A)$ is an $(\bar{i},\bar{j})$ irredundant digraph, then intervality of D can be checked in $O(n+a)$ time, where $n=|V|$ and $a=|A|$.*

Proof. Note that an $(\bar{i},\bar{j})$ irredundant competition graph G is an interval graph if and only if it is triangulated. To see why, note that every component of a triangulated graph has a simplicial vertex and therefore the $j(i-1)$-regularity of each component implies that each component is complete. It follows that we may use a simple variant of the Rose–Tarjan–Lueker [39] algorithm for obtaining a perfect elimination ordering for triangulated graphs to check for intervality of D. $\square$

Note that Theorem 3.20 does not give a forbidden subgraph characterization of $(\bar{i},\bar{j})$ irredundant interval digraphs.

4. Some open questions

In this paper, we have been studying the three basic questions of characterizing competition graphs, interval competition graphs, and interval digraphs, under various assumptions about D. We have solved these problems with one important exception when D is a $(2,2)$ digraph and when D is an $(\bar{i},\bar{j})$ irredundant digraph. Namely, our results leave open the question of characterizing the $(\bar{i},\bar{j})$ irredundant competition graphs in the general case. We also leave open in the general $(\bar{i},\bar{j})$ case the three basic questions we have been studying if we remove the assumption that digraphs must be irredundant. We do have some results for the $(\bar{2},\bar{2})$ case without irredundancy. The paper also leaves open all three basic questions for the general (i,j) case, even with irredundancy, and for the mixed cases $(\bar{i},j)$ and $(i,\bar{j})$. Finally, we leave open the question of whether there is a forbidden subgraph characterization of $(\bar{i},\bar{j})$ irredundant interval digraphs.

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